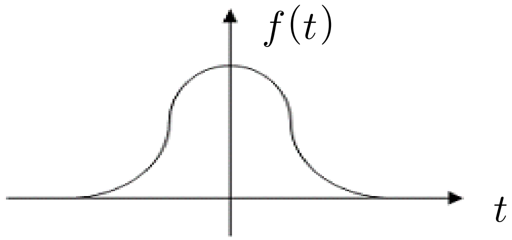
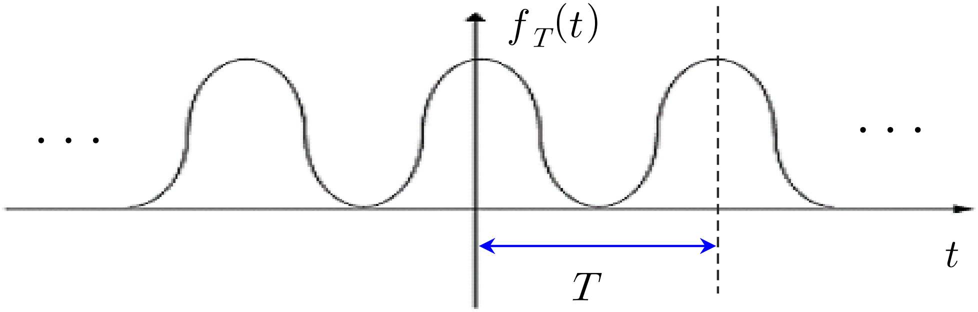
○ Fourier transform

Fourier series 전개에서 주기 $T \Rightarrow \uparrow$, $w_0 = \dfrac{2\pi}{T} \Rightarrow \downarrow$

Aperiodic Signal $f(t)$ $\rightarrow$



Periodic Signal $f_T(t)$ $\rightarrow$



$$\Rightarrow \; f(t) = \lim_{T \to \infty} f_T(t)$$

then, $$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jnw_0 t} \qquad \rightarrow ①$$

$$F_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-jnw_0 t} dt \qquad \rightarrow ②$$

$$w_0 = \frac{2\pi}{T} \qquad \rightarrow ③$$

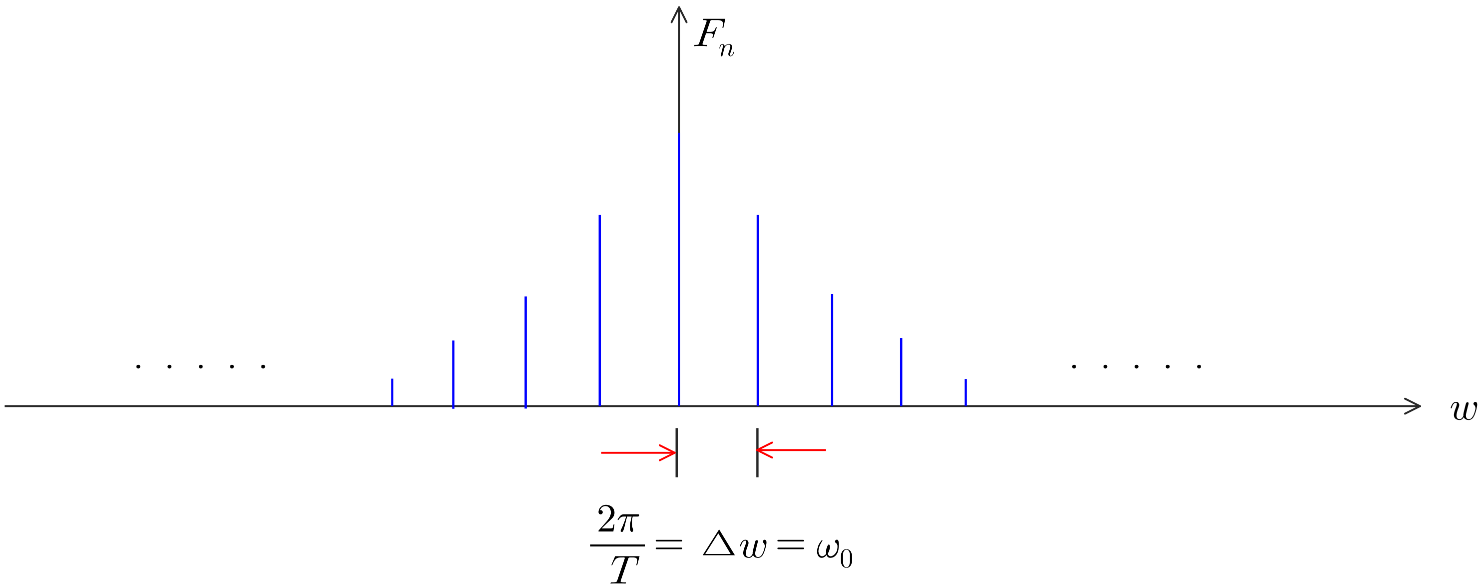
여기서 $T \longrightarrow \infty$ 로 보낼 때, F_n이 “0”이 되는 것을 방지하기 위해 $w_n \triangleq nw_0$, $F(w_n) \triangleq TF_n$ ⇒ 정의하면

식 ① & ②는 각각

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} F(w_n) e^{jw_n t} \qquad \rightarrow ④$$

$$F(\omega_n) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-jw_n t} dt \qquad \rightarrow ⑤$$

한편, Fourier 계수 F_n은 주파수 영역에서 Discrete하며 다음과 같은 line spectrum을 갖는다.



식 ④를 $\triangle w$를 이용해서 표현하면 $(\frac{1}{T}=\frac{\triangle w}{2\pi})$

$$f_T(t)=\sum_{n=-\infty}^{\infty}F(w_n)e^{jw_nt}\bullet\frac{\triangle w}{2\pi} \qquad \rightarrow ⑥$$

Then, $T\longrightarrow\infty$, $\triangle w\longrightarrow 0$ (meaning : spectrum이 continuous한 형태로 바뀐다.)

$T\longrightarrow\infty$로 보내면 식 ⑥은

$$\lim_{T\to\infty}f_T(t)=\lim_{T\to\infty}\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}F(w_n)e^{jw_nt}\triangle w \qquad \rightarrow ⑦$$

식 ⑦은 $T \longrightarrow \infty$에 따라 $\triangle w$가 dw가 됨. 그리고, spectrum이 continuous$(w_n \longrightarrow w)$함.

또한 $\sum_{n=-\infty}^{\infty} \longrightarrow \int_{-\infty}^{\infty}$ 가 됨. 결과적으로,

$$\lim_{T\to\infty} f_T(t) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{jwt}dw \quad \rightarrow ⑧$$

식 ⑤로부터 $w_n \longrightarrow w$, $T \longrightarrow \infty$, $\lim_{T\to\infty} f_T(t) = f(t)$를 대입하면

$$F(w) = \int_{-\infty}^{\infty} f(t)e^{-jwt}dt \quad \rightarrow ⑨$$

식 ⑧을 Inverse Fourier Transform (IFT)

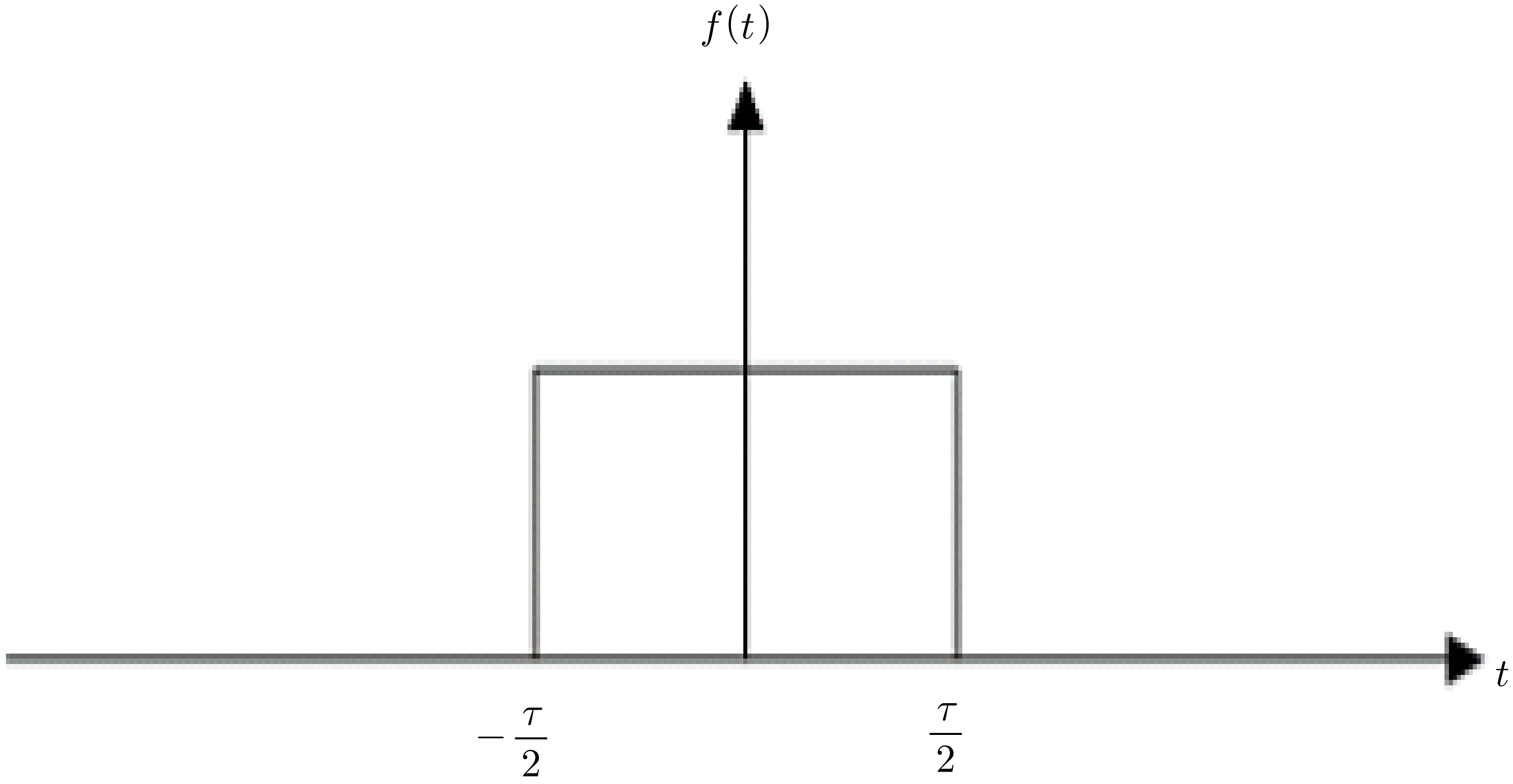
식 ⑨를 Foward Fourier Transform (FFT) 또는 FT이라 함.

▶중간정리◀

$$F(w) = \mathscr{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt$$: FT -시간영역에서 주파수영역으로

$$f(t) = \mathscr{F}^{-1}\{F(w)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w) e^{jwt} dw$$: IFT -주파수영역에서 시간영역으로

ex) Rectangular gate ft(width=τ)의 FT을 구하라.



$$F(w) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} (1) e^{-jwt} dt$$

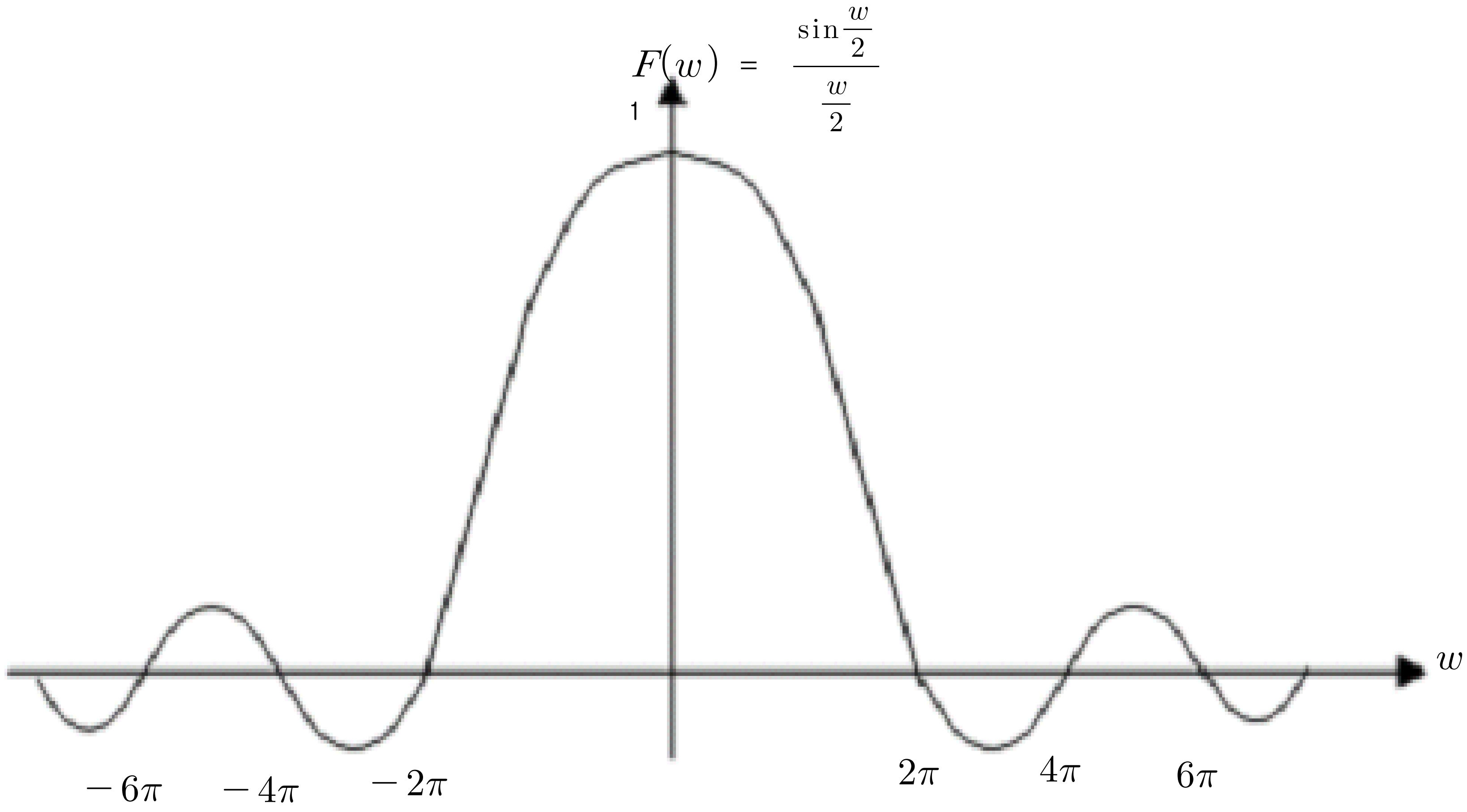
$$= \frac{1}{(-jw)} [e^{-jwt}]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{1}{(-jw)} [e^{-\frac{jw\tau}{2}} - e^{\frac{jw\tau}{2}}]$$

$$= \frac{1}{(-jw)} \{(\cos\frac{w\tau}{2} - j\sin\frac{w\tau}{2}) - (\cos\frac{w\tau}{2} + j\sin\frac{w\tau}{2})\}$$

$$= \frac{1}{(-jw)} (-2j\sin\frac{w\tau}{2}) = \frac{1}{w} (2\sin\frac{w\tau}{2})$$

$$= \boxed{\tau \bullet \frac{\sin\frac{w\tau}{2}}{\frac{w\tau}{2}}} = \tau \mathrm{sinc}(f\tau) \longrightarrow \frac{\sin\pi f\tau}{\pi f\tau}$$ if τ=1, f=1→"0"

if $\tau = 1$



○ Parseval's theorem

Enegy $E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) \bullet f^*(t) dt \Leftarrow$ $|a+bj|^2 = (\sqrt{a^2+b^2})^2$

$= (a+bj)(a-bj)$

$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{jwt} dw$를 대입하면

$E = \int_{-\infty}^{\infty} f(t) [\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(w) e^{-jwt} dw] dt$ $\leftarrow dt,\ dw$의 순서를 바꾸면

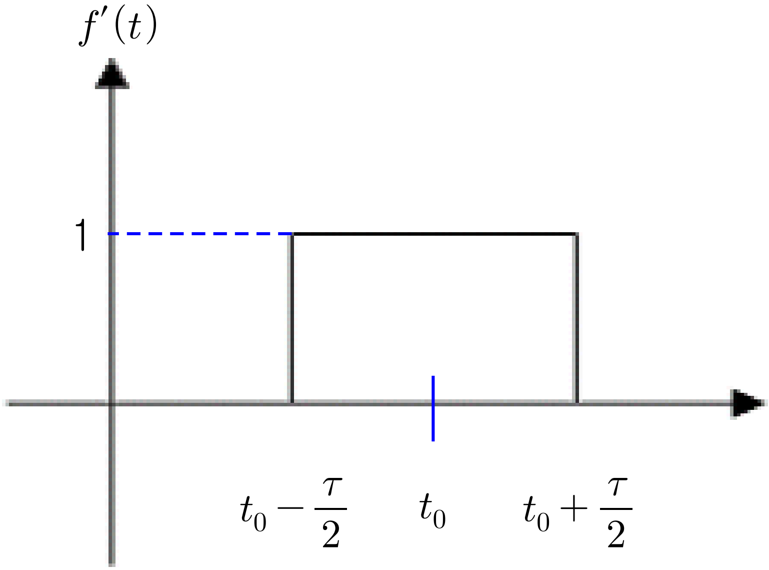
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(w)[\int_{-\infty}^{\infty} f(t)e^{-jwt}dt]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(w) \bullet F(w)dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(w)|^2 dw$$

$*$ → Conjugate

주파수 영역의 Energy와 같다.

○ FT의 기본 성질(time shift)



$$F'(w) = \int_{t_0 - \frac{\tau}{2}}^{t_0 + \frac{\tau}{2}} e^{-jwt}\, dt$$

$$= \frac{1}{-jw}\left[e^{-jw(t_0+\frac{\tau}{2})} - e^{-jw(t_0-\frac{\tau}{2})}\right]$$

$$= \frac{1}{-jw}\left[e^{-jwt_0}(e^{\frac{-jw\tau}{2}} - e^{\frac{jw\tau}{2}})\right]$$

$$= \frac{e^{-jwt_0}}{-jw}(-2j\sin\frac{w\tau}{2})$$

$$= \tau\frac{\sin\frac{w\tau}{2}}{\frac{w\tau}{2}}e^{-jwt_0}$$

$$= \mathscr{F}\{f(t)\}e^{-jwt_0} \qquad \Rightarrow \qquad \text{complex}$$

· $f(t)$가 y축에 대해 대칭이면 $F(w)$는 Real ft

· $f(t)$가 y축에 대해 비대칭(time shift)이면 $F(w)$는 Complex ft

· $F(w)$가 y축에 대해 대칭(complex일 경우 conjugate)이면

$f(t)$는 Real ft

· $F(w)$가 y축에 대해 비대칭(complex일 경우 conjugate이 아니면)이면

$f(t)$는 Complex ft

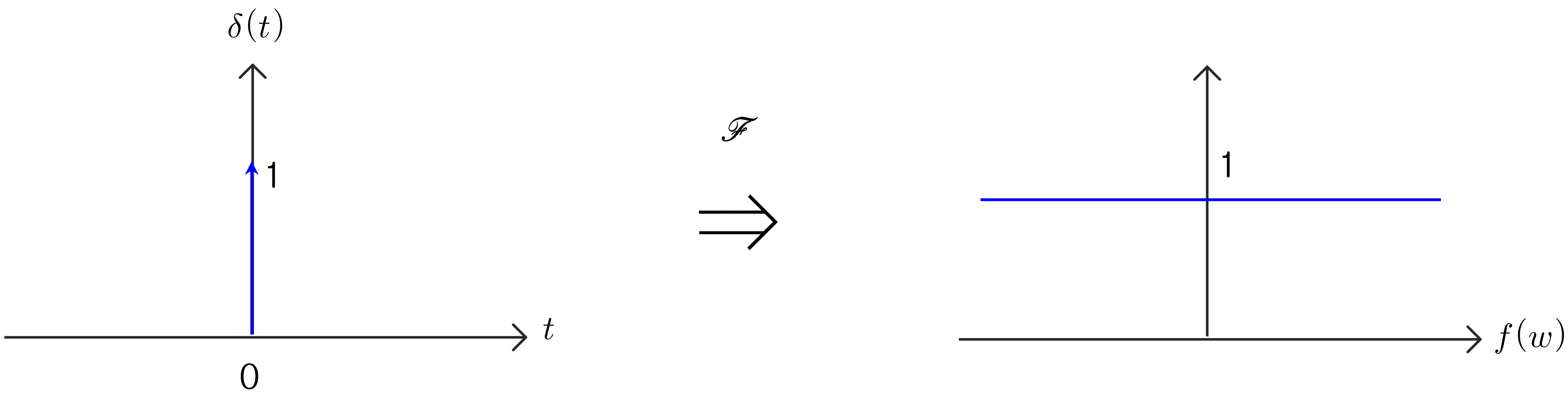
○ Special ft에 대한 FT

① Impulse ft

$$\mathscr{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \underline{\delta(t)e^{-jwt}}dt = e^{-j0} \bullet \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

↑

$t=0$일경우만 값이 존재



② Complex exponential ($e^{\pm jw_0 t}$)

$$\mathscr{F}^{-1}\{\delta(w \mp w_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(w \mp w_0)\, e^{jwt}\, dw = \frac{1}{2\pi}\, e^{\pm jw_0 t}$$

위 식의 양변에 FT를 하면

$$\underline{\mathscr{F}\mathscr{F}^{-1}\{\delta(w \mp w_0)\}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(w \mp w_0) e^{jwt} dw = \frac{1}{2\pi}\mathscr{F}\{e^{\pm jw_0 t}\}$$

$$\{\delta(w \mp w_0)\}$$

$$\therefore\ \mathscr{F}\{e^{\pm jw_0 t}\} = 2\pi\delta(w \mp w_0)$$

③ Sinusoidal ft. $\left(\cos w_0 t\,,\ \sin w_0 t\right)$

$$\mathscr{F}\{\cos w_0 t\} = \mathscr{F}\{\frac{1}{2}e^{jw_0 t} + \frac{1}{2}e^{-jw_0 t}\}$$

$$= \pi\delta(w - w_0) + \pi\delta(w + w_0)$$

$$\mathscr{F}\{\sin w_0 t\} = \mathscr{F}\{\frac{1}{2j}\{e^{jw_0 t} - e^{-jw_0 t}\}\}$$

$$= \frac{1}{j}[\pi\delta(w - w_0) - \pi\delta(w + w_0)]$$

○ FT 의 성질

① Linearity

$$\mathscr{F}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(w) + a_2 F_2(w)$$

② Complex conjugate

$$\mathscr{F}\{f^*(t)\} = \int_{-\infty}^{\infty} f^*(t) e^{-jwt} dt = \underbrace{[\int_{-\infty}^{\infty} f(t) e^{jwt} dt]}_{F(-w)}{}^*$$

$$= F^*(-w) \quad \Rightarrow \quad F(w) = F^*(-w)$$ 이면 f(t)는 실수

③ Symmetry

임의의 함수 $f(t)$는 짝함수와 홀함수의 합으로 표현이 가능하다.

$$f(t) = f_e(t) + f_o(t)$$

$$\mathscr{F}\{f(t)\} = \int_{-\infty}^{\infty} (f_e(t) + f_o(t))e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} f_e(t)\,e^{-jwt}dt + \int_{-\infty}^{\infty} f_0(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} f_e(t)(\cos wt - j\sin wt)dt + \int_{-\infty}^{\infty} f_0(t)(\cos wt - j\sin wt)dt$$

$$= \int_{-\infty}^{\infty} (f_e(t)\cos wt - jf_e(t)\sin wt) + (f_0(t)\cos wt - jf_0(t)\sin wt)dt$$

$$= \int_{-\infty}^{\infty} (f_e(t)\cos wt - jf_0(t)\sin wt)dt$$

$$= 2\int_{0}^{\infty} (f_e(t)\cos wt - jf_0(t)\sin wt)dt$$

$\therefore\ \mathscr{F}\{f_e(t)\} = F_e(w) =$ Real$\{F(w)\}$, 실수값

$\mathscr{F}\{f_0(t)\} = F_0(w) =$ Imag$\{F(w)\}$, 허수값

④ Duality

if $\mathscr{F}\{f(t)\} = F(w)$

정의식으로부터 $F(w) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt$ → ①

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{jwt} dw$$ → ②

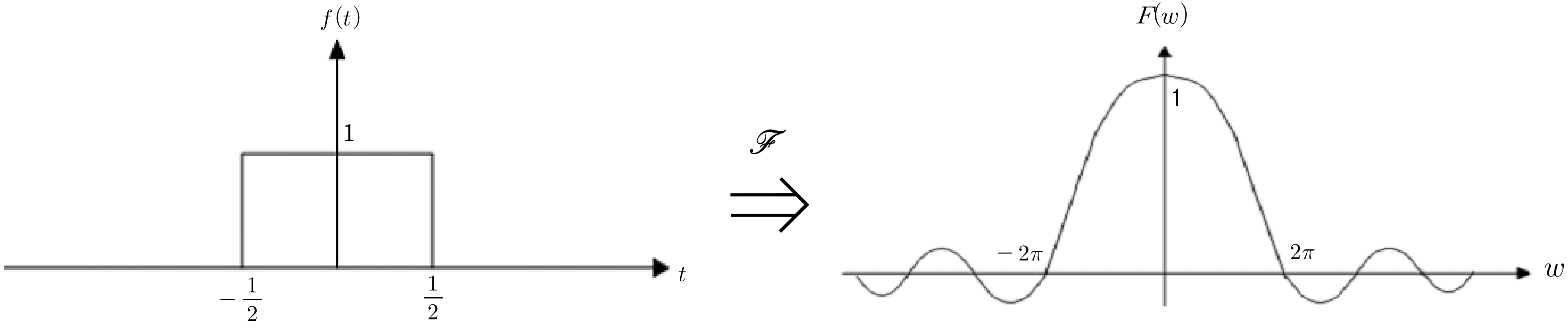
$$\mathscr{F}\{F(t)\} = \int_{-\infty}^{\infty} F(t) e^{-jwt} dt$$ → ③

식 ②로부터 w를 t로, t를 $-w$로 바꾸면

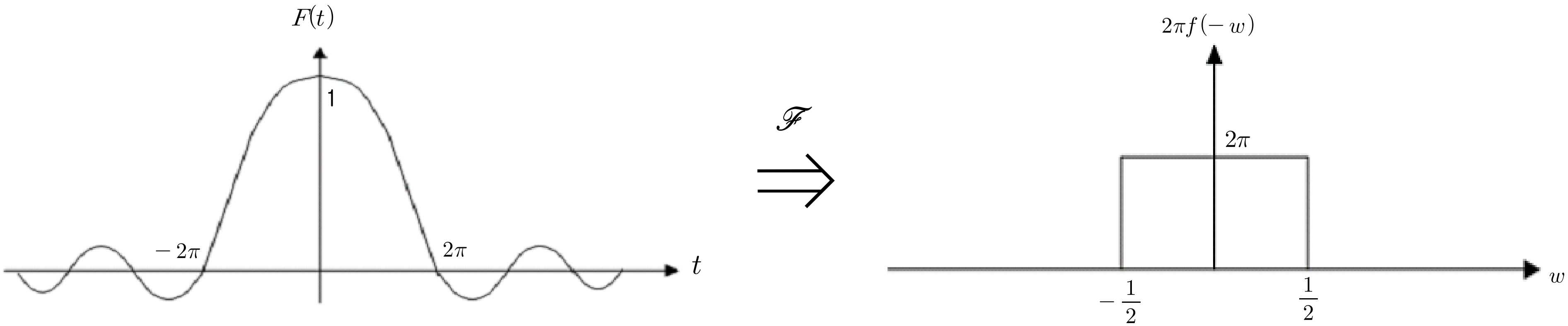
$$f(-w) = \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} F(t)e^{-jwt}dt}_{③번\ 식} = \frac{1}{2\pi}\mathscr{F}\{F(t)\}$$

$$\therefore \boxed{\mathscr{F}\{F(t)\} = 2\pi f(-w)}$$

ex)



$f(t)$
1
$-\frac{1}{2}$
$\frac{1}{2}$
t
$\mathscr{F}$
$\Rightarrow$
$F(w)$
1
-2π
2π
w



$F(t)$
1
-2π
2π
t
$\mathscr{F}$
$\Rightarrow$
$2\pi f(-w)$
2π
$-\frac{1}{2}$
$\frac{1}{2}$
w

⑤ Time Shifting & Frequency Shifting

-Time shifting $\mathscr{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0)e^{-jwt}dt$, Let $x = t - t_0$

$$= \int_{-\infty}^{\infty} f(x)e^{-jw(x+t_0)}dx = e^{-jwt_0}\underbrace{\int_{-\infty}^{\infty} f(x)e^{-jwx}dx}_{F(w)}$$

$$= e^{-jwt_0}F(w)$$

⇒ $f(t)$가 시간영역에서 t_0만큼 Delay되면, 주파수 영역에서는 크기는 변화가 없고

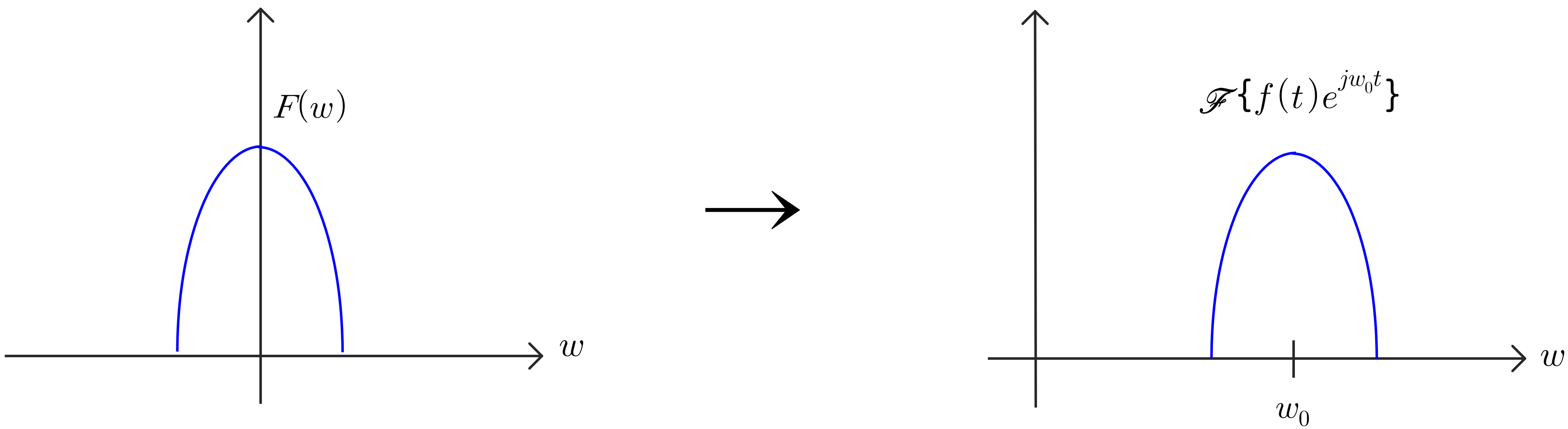
Phase만 $(-wt_0)$만큼 변화한다.

- Frequency Shifting=Modulation(변조)

$$\mathscr{F}\{f(t)e^{jw_0t}\} = \int_{-\infty}^{\infty}\{f(t)e^{jw_0t}\}e^{-jwt}dt = \int_{-\infty}^{\infty} f(t)e^{-j(w-w_0)t}dt$$

Modulation
(Frequency shifting)

$$= F(w-w_0)$$



※ Modulation을 사용하는 가장 중요한 이유

⇒ 안테나의 크기를 줄일 수 있어, 이동성이 증가한다. (저주파 → 안테나 大, 고주파 → 안테나 小)

⑥ Differentiation

$$\mathscr{F}\{\frac{d}{dt}f(t)\} \rightarrow jwF(w)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{jwt}dw$$

$$\frac{d}{dt}f(t) = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty} F(w)e^{jwt}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\frac{d}{dt}[e^{jwt}]dw$$

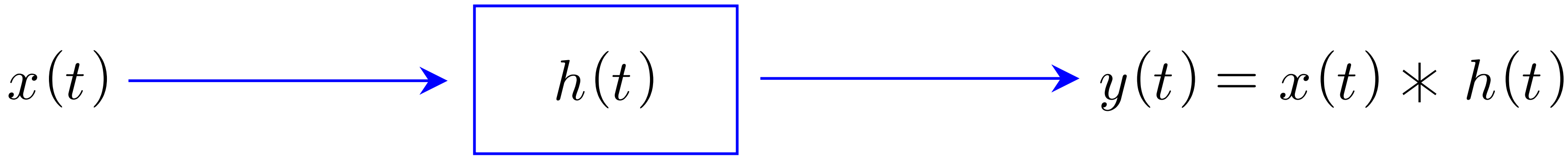
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}[jwF(w)]e^{jwt}dw \quad \Rightarrow \quad jwF(w)$$ 함수의 IFT 이다.

$$= \mathscr{F}^{-1}[jwF(w)]$$

양변에 $\mathscr{F}$을 취하면

$$\mathscr{F}\{\frac{d}{dt}f(t)\} = \mathscr{F}\mathscr{F}^{-1}[jwF(w)] = jwF(w)$$

⑦ Convolution



$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$\mathscr{F}\{x(t) * h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right]e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-jwt}dt\right]d\tau$$

위의 식에서 time shift 성질을 이용하면

$$= \int_{-\infty}^{\infty} x(\tau) \bullet [e^{-jw\tau} \bullet H(w)] d\tau$$

$$= H(w) \int_{-\infty}^{\infty} x(\tau) e^{-jw\tau} d\tau$$

$$= H(w) X(w) = X(w) H(w)$$

따라서, Time domain의 $\ast$은 Frequency domain의 $\times$

Duality 성질에 따라

Frequency domain의 $\ast$은 Time domain에서 $\times$이다.